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Dynamics of Mean-Field Spin Models from Basic Results in Abstract Differential Equations

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The infinite-volume limit of the dynamics of (generalized) mean-field spin models is obtained through a direct analysis of the equations of motion, in a large class of representations of the spin algebra. The resulting dynamics fits into a general framework for systems with long-range interaction: variables at infinity appear in the time evolution of local variables and spontaneous symmetry breaking with an energy gap follows from this mechanism. The independence of the construction of the approximation scheme in finite volume is proven.

KEY WORDS: Mean-field spin models; infinite-volulme limit; dynamics of systems with long-range interactions; dynamics out of equilibrium; differential equations in C^* and von Neumann algebras; spontaneous symmetry breaking; generalized Goldstone's theorem; KMS condition and long-range interactions.

INTRODUCTION

Mean-field spin models (MFSM) play an important role in quantum statistical mechanics (QSM) both for their phenomenological relevance (see the "spin" formulation of the BCS model in refs. 1, 2) and as "exactly solvable" models showing general characteristic phenomena of QSM. The construction of the dynamics of MFSM does not fit, however, into the standard treatment of spin models (see, e.g., ref. 3); in fact, due to the long-range and volume dependence of the mean-field interaction, the infinite-volume limit of the dynamics cannot be controlled as in the case of short-range interactions,[4] and it does not give to a well-defined transformation of the local (spin) variables into themselves.[5,6]

In such a situation, the construction of the dynamics in infinite

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volume can be based on two different strategies: (i) one essentially .takes the limit of correlation functions (at all times) for certain sequences of (time-translation invariant) finite-volume states, time evolution being defined in finite volume[2,7–9]; (ii) for a class of representations (of the quasilocal algebra) one takes the strong limit, for volumes going to infinity, of the time evolution defined in finite volumes;[5,10–12] here the states do not depend on the volume and only enter to define the notion of convergence of algebraic objects.

Strategy (i) is limited to a restricted class of representations of the spin algebra, typically those defined by ground or finite-temperature states. This limitation, the strong dependence of the construction on the representation, and therefore the absence of an algebraic formulation of the models in infinite volume, lead in particular to:

1. The impossibility of discussing the time evolution of classical (mean) variables out of thermal equilibrium.

2. The impossibility of formulating the KMS consition as a restriction which identifies the thermal states on the basis of a given dynamics.

3. The loss from the beginning, through the choice of the representation of the spin algebra, of the symmetries of the Hamiltonian, so that questions about spontaneous symmetry breaking cannot even be posed.

As a consequence, the implications of MFSM as prototypes of systems with long-range interactions, in particular for the mechanism of symmetry breaking with an energy gap, are not clear in this approach.

The second strategy has been applied to the mean-field Ising model in ref. 5 and to a class of "generalized Ising" mean-field models in ref. 13. The construction of the dynamics of general MFSM has been given in ref. 11. The purpose of the present paper is to present the proof of the results of ref. 11 in a slightly generalized version; we also discuss their implications for the general problems of the dynamics of systems with long-range interactions and for the KMS condition, proving that for general MFSM, strategy (i) gives rise to the restriction to a special class of representations of the dynamics resulting from strategy (ii).

The construction of the algebraic dynamics for (generalized) MFSM has also been performed in refs. 14 and 15 with different methods (for related results see refs. 16–20). A construction of the infinite-volume dynamics of MFSM in a spirit similar to ours has been given in ref. 21 as an application of rather sophisticated mathematical structures (quasi-*-algebras and topologies for unbounded operators) which do not seem to us to be the most appropriate for MSFM, where only bounded variables are involved.

The infinite-volume limit of the dynamics is obtained here directly from the analysis of the equations of motion of spin variables in finite volumes. This analysis only requires almost straightforward generalizations of elementary results on differential equations with analytical coefficients (in a spirit similar to that of Section 2 of ref. 22); it also applies to a wider class of models, essentially whenever variables at a lattice point are only coupled to a fixed number of variables, which may depend on the volume but have a well-defined infinite-volume limit.

The paper is organized as follows: In Section 1, the problem is formulated and the strategy explained. In Section 2, the results are shown to follow from abstract results about differential equations for variables in C^* algebras. Section 3 contains the analysis of abstract differential equations. In Section 4, we prove that our construction of the dynamics gives the same result as the one based on the convergence of correlation functions, for a large class of representations of the quasilocal (spin) algebra; we then point out some implications on the structural properties of the dynamics of systems with long-range interactions, the KMS condition, and spontaneous symmetry breaking with an energy gap.

1. MODELS AND STRATEGY

Mean-field spin models are defined by a state space $\mathbb{C}^2$ at each site of a lattice $\mathbb{Z}^d$ and by the finite-volume Hamiltonians

$$H_V = \frac{1}{|V|} \sum_{i,j \in V} \sum_{\alpha,\beta=1,2,3} A^{\alpha\beta} \sigma_i^\alpha \sigma_j^\beta + \sum_{i \in V} \sum_{\alpha=1,2,3} C^\alpha \sigma_i^\alpha \tag{1.1}$$

where $i, j \in \mathbb{Z}^d$ (the dimensional d playing a minor role), σ_i^α are the Pauli matrices acting in the two-dimensional space at site i, V is any finite subset of $\mathbb{Z}^d$, $|V|$ is the number of its points; $A^{\alpha\beta}$ is a Hermitian matrix and C^α are real numbers, so that H_V is Hermitian.

For $A^{\alpha\beta} = \delta^{\alpha\beta}$, the Hamiltonian (1.1) defines the Heisenberg–Weiss model; for

$$A^{\alpha\beta} = -\frac{T_c}{2} \begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad C^\alpha = -\varepsilon \delta^{\alpha 3}$$

It gives the Hamiltonian of the BCS model in the lattice formulation of Thirring.[1,2]

Our treatment applies to a larger class of models, defined by a state space $\mathscr{H}_i$ at each site i of $\mathbb{Z}^d$, which $\mathscr{H}_i = U_i \mathscr{H}$, U_i isometric, and by finite-volume Hamiltonians

$$H_V = \frac{1}{|V|^{k-1}} \sum_{i_1 \cdots i_k \in V} \sum_{\alpha_1 \cdots \alpha_k} A^{\alpha_1 \cdots \alpha_k} M_{i_1}^{\alpha_1} \cdots M_{i_k}^{\alpha_k} \tag{1.2}$$

where V is a finite subset of $\mathbb{Z}^d$ and $|V|$ its cardinality, $M_i^\alpha = U_i M^\alpha U_i^{-1}$, M^α a linear basis for the algebra of $n \times n$ matrices, $M^{\alpha\dagger} = M^\alpha$; $A^{\alpha_1 \cdots \alpha_k} = \bar{A}^{\alpha_k \cdots \alpha_1}$, so that H_V is Hermitian. The Hamiltonians (1.1) and (1.2) have no meaning for $|V| = \infty$ and our scope is to construct the dynamics in infinite volume as a limit of the Heisenberg dynamics in finite volumes.

Let $\mathscr{A}_V$ be the C^* algebra generated by the matrices M_i^α, $i \in V$, and $\mathscr{A}$ the norm-closure of $\bigcup_V \mathscr{A}_V$. Since H_V is in $\mathscr{A}_V$, the Hamiltonians (1.2) define a one-parameter group of automorphisms of $\mathscr{A}$ by

$$A \mapsto \alpha_V^t(A) \equiv \exp(iH_V t) A \exp(-iH_V t) \tag{1.3}$$

For lattice models with short-range interactions the limit for $V \to \infty$ of $\alpha_V^t(A)$ exists in the norm topology of $\mathscr{A}$ and defines a group of automorphisms of $\mathscr{A}$ (see ref. 7). Norm convergence does not hold, however, for the dynamics defined by the Hamiltonians (1.1), (1.2), as one can see immediately from the solution of special models[5]; the point is that the solution involves the operators $\sigma_V^\alpha \equiv (1/|V|) \sum_{i \in V} \sigma_i^\alpha$, which do not define a Cauchy sequence in the norm topology.

Following the strategy (ii) mentioned above, and in particular the general scheme developed in refs. 11, 12, we will show that the finite-volume Heisenberg dynamics $\alpha_V^t(A)$ converges, for each $A \in \mathscr{A}$, in the ultrastrong topology defined by a class $\mathscr{F}^+$ of states over $\mathscr{A}$, and that the limit defines a group of ultrastrongly continuous automophisms of the strong closure of $\mathscr{A}$. By $V \to \infty$ we mean that we take an increasing sequence $\{V_n\}$ of subsets of $\mathbb{Z}^d$ such that each lattice point i belongs to V_n for $n \geqslant n(i)$; some regularity condition may be assumed on the sequence of volumes (e.g., one may consider only cubes), and all the results are independent of such conditions, in the sense that any change in the notion of convergence is only reflected in the identification of the set $\mathscr{F}^+$ of states which define the relevant (ultra-) strong topology.

Technically, we shall consider a class $\mathscr{F}^+$ of states, i.e., positive linear functionals, over the C^* algebra $\mathscr{A}$, which is the positive part of a set $\mathscr{F}$ of continuous linear functionals over $\mathscr{A}$ with the following properties:

F1: $\mathscr{F}$ is closed under linear combinations and norm limits.

F2: if $\omega(\cdot) \in \mathscr{F}$ and $A, B \in \mathscr{A}$, then $\omega(A \cdot B) \in \mathscr{F}$.

A set of states with the above properties is exactly the sets of normal states in some family of representations of $\mathscr{A}$ (ref. 12, Proposition 2.1), and therefore a "full folium" in the sense of ref. 23.

We shall first construct $\mathscr{F}^+$ as the largest set of states such that the space ergodic means of the matrix variables M_i^α exist in the (ultra-) strong topology defined by them; this set will turn out to be the largest for which α_V^t converges ultrastrongly [for all Hamiltonians of the form (1.2)].

We shall then analyze the equations of motion satisfied by $\alpha_V^t(M_i^\alpha)$, and reduce the $\mathscr{F}$-(ultra-) strong convergence of $\alpha_V^t(M_i^\alpha)$ to a problem of dependence upon the initial conditions of (a system of) differential equations with analytical coefficients for variables in a C^* algebra. As a result, strong convergence of space ergodic means of M_i^α will imply strong convergence of $\alpha_V^t(M_i^\alpha)$.

Strong convergence of $\alpha_V^t(A)$ will follow for any $A \in \mathscr{A}$, and the limit of α_V^t will define a group of strongly continuous automorphisms of the algebra generated by $\mathscr{A}$ and by the ergodic mean variables

$$\mathscr{F}\text{-ultrastrong-}\lim_{V \to \infty} \frac{1}{|V|} \sum_{i \in V} M_i^\alpha \tag{1.4}$$

The abstract results on analyticity and dependence upon initial values for differential equations in C^* algebras are rather straightforward generalizations of well-known results and will be proven in Section 3.

2. THE INFINITE-VOLUME LIMIT OF THE DYNAMICS

2.1. The Class of Relevant States

A class $\mathscr{F}^+$ of states is defined as follows: Given a state (i.e., a positive linear functional) ω over $\mathscr{A}$, denote by $\mathscr{H}_\omega$ the Hilbert space defined by the Gelfand–Naimark–Segal (GNS) construction on $\mathscr{A}$ and ω, and by ψ_ω the vector which represents ω in $\mathscr{H}_\omega$: $\omega(A) = (\psi_\omega, A\psi_\omega)$. Call a state ω admissible if

$$\lim_{V \to \infty} \frac{1}{|V|} \sum_{i \in V} M_i^\alpha \psi_\omega \tag{2.1}$$

exists in the strong Hilbert space topology of $\mathscr{H}_\omega$. Notice that, for uniformly bounded sequences of operators, the strong and ultrastrong topologies coincide (see, e.g., ref. 24); we will therefore simply refer to the strong topology here and in the following.

Definition 2.1. Let $\mathscr{F}^+$ be the set of all admissible states and $\mathscr{F}$ the linear space generated by $\mathscr{F}^+$.

Proposition 2.2.[11] $\mathscr{F}$ satisfies properties F1 and F2 of Section 1; equivalently, $\mathscr{F}^+$ is a full folium in the sense of ref. 23.

Proof. F1: Strong convergence of (2.1) is equivalent to

$$\omega\left(\left(\frac{1}{|V|}\sum_{i\in V} M_i^\alpha - \frac{1}{|V'|}\sum_{i\in V'} M_i^\alpha\right)^2\right) \to 0 \qquad \text{as } V \text{ and } V' \to \infty \tag{2.2}$$

and, by uniform boundedness in norm of the operators, if (2.2) holds for a norm-convergent sequence ω_n, it also holds for its limit.

F2: If $A \in \mathscr{A}_{V_0}$ for some V_0, then, due to the commutation of variables at different sites,

$$\frac{1}{|V|}\sum_{i\in V} M_i^\alpha A\psi_\omega = A\frac{1}{|V|}\sum_{i\in V} M_i^\alpha \psi_\omega + \frac{1}{|V|}\sum_{i\in V_0} [M_i^\alpha, A]\psi_\omega \tag{2.3}$$

The norm of the second term in the rhs is of order $1/|V|$, so that, if ω is admissible, $A\psi_\omega$ represents an admissible state, i.e., the state $\omega(A^\dagger \cdot A)$ is admissible. The same holds for any A in $\mathscr{A}$, by norm limits of states.

The result for $\omega(A^\dagger \cdot B)$, $\forall A, B \in \mathscr{A}$, follows immediately by writing this linear functional as a sum of states of the form $\omega(C^\dagger \cdot C)$. QED

By Proposition 2.2, strong convergence (with a fixed meaning for $V \to \infty$) of ergodic means of the variables M_i^α identifies a class of linear functionals $\mathscr{F}$ with the properties F1, F2; its positive part $\mathscr{F}^+$ will be called "the family of relevant states." $\mathscr{F}^+$ (which has a dependence upon the notion of infinite-volume limit) always contains all translation-invariant product states, i.e., states ω such that, for different lattice indices $i_1 \cdots i_k$ and $\forall i$,

$$\omega(M_{i_1}^{\alpha_1} \cdots M_{i_k}^{\alpha_k}) = \omega(M_i^{\alpha_1}) \cdots \omega(M_i^{\alpha_k})$$

A more general result is the following.

Proposition 2.3. Let ω be invariant under the automorphisms of $\mathscr{A}$ defined by all exchanges of lattice points:

$$\omega(M_{i_1}^{\alpha_1} \cdots M_{i_k}^{\alpha_k}) = \omega(M_{j_1}^{\alpha_1} \cdots M_{j_k}^{\alpha_k})$$

$\forall i_1 \cdots i_k$ all different and $\forall j_1 \cdots j_k$ all different. Then ω is in $\mathscr{F}^+$ (for any notion of infinite-volume limit; see Definition 2.1 and Section 1).

Proof. As in Proposition 2.1, we must prove

$$\omega\left(\left(\frac{1}{|V|}\sum_{i\in V} M_i^\alpha - \frac{1}{|V'|}\sum_{i\in V'} M_i^\alpha\right)^2\right) \to 0 \qquad \text{as } V \text{ and } V' \to \infty$$

By expanding the sums in the above expression and separating terms at the same lattice point, one obtains, with $V \subset V'$,

$$\left(\sum_{i \in V} \frac{1}{|V|^2} + \sum_{i \in V'} \frac{1}{|V'|^2} - 2 \sum_{i \in V} \frac{1}{|V|} \frac{1}{|V'|}\right) \omega(M_i^\alpha M_i^\alpha)$$

$$+ \left(\sum_{\substack{i,j \in V \\ i \neq j}} \frac{1}{|V|^2} + \sum_{\substack{i,j \in V' \\ i \neq j}} - 2 \sum_{\substack{i \in V, j \in V' \\ i \neq j}} \frac{1}{|V|} \frac{1}{|V'|}\right) \omega(M_i^\alpha M_j^\alpha)$$

$$= \left(\frac{1}{|V|} - \frac{1}{|V'|}\right) [\omega(M_i^\alpha M_i^\alpha) - \omega(M_i^\alpha M_j^\alpha)] \to 0$$

as $V, V' \to \infty$. QED

2.2. The Equations of Motion and the Limit of the Dynamics as $V \to \infty$

Given a finite-volume Hamiltonian of the form (1.2), the time evolution in finite volume of the variables M_i^α satisfies the differential equations

$$\frac{d}{dt} \alpha_V^t(M_i^\alpha) = \frac{i}{|V|^{k-1}} \sum_{i_1 \cdots i_k \in V} \sum_{\alpha_1 \cdots \alpha_k} A^{\alpha_1 \cdots \alpha_k} \alpha_V^t([M_{i_1}^{\alpha_1} \cdots M_{i_k}^{\alpha_k}, M_i^\alpha])$$

$$= \frac{i}{|V|^{k-1}} \sum_{i_1 \cdots i_k \in V} \sum_{\alpha_1 \cdots \alpha_k} A^{\alpha_1 \cdots \alpha_k} \sum_{j=1 \cdots k} C_{\alpha_j \alpha \beta}$$

$$\times \alpha_V^t(M_{i_1}^{\alpha_1} \cdots M_{i_{j-1}}^{\alpha_{j-1}} M_i^\beta M_{i_{j+1}}^{\alpha_{j+1}} \cdots M_{i_k}^{\alpha_k})$$

$$= i \sum_{\alpha_1 \cdots \alpha_k} A^{\alpha_1 \cdots \alpha_k} \sum_{j=1 \cdots k} C_{\alpha_j \alpha \beta}$$

$$\times \alpha_V^t(M_V^{\alpha_1}) \cdots \alpha_V^t(M_V^{\alpha_{j-1}}) \, \alpha_V^t(M_i^\beta) \, \alpha_V^t(M_V^{\alpha_{j+1}}) \cdots \alpha_V^t(M_V^{\alpha_k}) \quad (2.4)$$

with $C_{\alpha\beta\gamma}$ defined by

$$[M_i^\alpha, M_i^\beta] = \sum_\gamma C_{\alpha\beta\gamma} M_i^\gamma$$

and

$$M_V^\alpha \equiv \frac{1}{|V|} \sum_{i \in V} M_i^\alpha$$

The equations of motion for the variables M_V^α in volume V follow immediately from Eq. (2.4), by summing over $i \in V$:

$$\frac{d}{dt}\alpha_V^t(M_V^\alpha) = i \sum_{\alpha_1 \cdots \alpha_k} A^{\alpha_1 \cdots \alpha_k} \sum_{j=1 \cdots k} C_{\alpha_j \alpha \beta} \times \alpha_V^t(M_V^{\alpha_1}) \cdots \alpha_V^t(M_V^{\alpha_j - 1})\, \alpha_V^t(M_V^\beta)\, \alpha_V^t(M_V^{\alpha_j + 1}) \cdots \alpha_V^t(M_V^{\alpha_k}) \tag{2.5}$$

Equations (2.4) and (2.5) form a closed system of differential equations for the variables M_i^α (with i a fixed lattice point) and M_V^β; introducing vectors x_V^t and y_V^t given by $x_V^t \equiv \alpha_V^t(M_i^\alpha)$ and $y_V^t \equiv \alpha_V^t(M_V^\beta)$, we find that Eqs. (2.4) and (2.5) take the form

$$\frac{d}{dt}x_V^t = f(x_V^t, y_V^t), \qquad \frac{d}{dt}y_V^t = g(y_V^t) \tag{2.6}$$

The crucial point is that the (vector) functions f and g which appear in Eq. (2.6) are independent of V and polynomial in their variables.

In Section 3 we shall prove (Theorems 3.4, 3.5) and equations of the form

$$\frac{d}{dt}X_t = F(X_t) \tag{2.7}$$

with X_t a vector with components in a C^* algebra and F analytic have (for small $|t|$) a unique solution, which is an analytic function of the Cauchy data X_0, and converges ultrastrongly (with respect to any given ultrastrong topology) when X_0 does. From these results we get:

Theorem 2.4.[11] The finite-volume dynamics $\alpha_V^t(A)$ defined by the Hamiltonians (1.1) or (1.2) converges in the ultrastrong topology defined by the set of states $\mathscr{F}^+$ introduced in Definition 2.1, for any $A \in \mathscr{A}$, to a group of ultrastrongly continuous automorphisms α^t of the algebra generated (by sums, products, and norm-limits) by $\mathscr{A}$ and by the ergodic means

$$\mathscr{F}\text{-ultrastrong-}\lim_{V \to \infty} \frac{1}{|V|} \sum_{i \in V} M_i^\alpha$$

Proof. 1. The existence of the ultrastrong limit

$$\lim_{V \to \infty} \alpha_V^t(M_i^\alpha) \equiv \alpha^t(M_i^\alpha)$$

follows, for $|t|$ small enough, from Theorems 3.4 and 3.5: in fact, $\alpha_V^t(M_i^\alpha)$ is the solution (Theorem 3.4) of a system of equations of the form (2.6), with f and g polynomials and initial data $x_V^0 = M_i^\alpha$, $y_V^0 = M_V^\alpha$; since M_V^α converges ultrastrongly as $V \to \infty$ by definition of $\mathscr{F}$, the solution converges ultrastrongly (Theorem 3.5). Since (ultra-) strong convergence commutes with sums and products, $\alpha_V^t(A)$ converges ultrastrongly for any $A \in \bigcup_V \mathscr{A}_V$; by combining strong convergence with a norm limit, it follows immediately that $\alpha_V^t(A)$ converges ultrastrongly for any A in $\mathscr{A}$ and the limit $\alpha^t(A)$ preserves sums, products, and * operation.

2. By Theorem 3.5, $\alpha^t(M_i^\alpha)$ satisfies the differential equations

$$\frac{d}{dt}\alpha^t(M_i^\alpha) = f^\alpha(\alpha^t(M_i^\beta), \alpha^t(M_\infty^\gamma)) \tag{2.8}$$

and

$$\frac{d}{dt}\alpha^t(M_\infty^\alpha) = g^\alpha(\alpha^t(M_\infty^\beta)) \tag{2.9}$$

with f and g defined by Eqs. (2.6) and

$$M_\infty^\alpha \equiv \lim_{V \to \infty} \frac{1}{|V|} \sum_{i \in V} M_i^\alpha \tag{2.10}$$

From Theorem 3.4 it follows that $\alpha^t(M_i^\alpha)$ is an analytic function, i.e. (Section 3), a norm-limit of polynomials, of the variables M_i^β and M_∞^γ; α^t therefore maps M_i^α into the algebra generated (by linear combinations, products, and norm-closure) by the matrix algebra at the site i and by the ergodic means (2.10). The algebra $\mathscr{A}$ is therefore mapped by α^t into the algebra generated by $\mathscr{A}$ and the algebra $\mathscr{A}_\infty$ generated by the ergodic means (2.10).

3. By Eq. (2.8) and Theorem 3.4, $\alpha^t(M_V^\alpha)$ is an analytic function of M_V^β and M_∞^γ and therefore it converges ultrastrongly as $V \to \infty$. It follows that the family $\mathscr{F}^+$ of relevant states is stable under the (transpose of the) time evolution α^t and therefore (see, e.g., Proposition 2.2 in ref. 12) α^t is continuous in the ultrastrong topology defined by $\mathscr{F}^+$. In particular, the time evolution of an ergodic mean (2.10) coincides with the ergodic mean of the time evolution; this also follows directly from Eqs. (2.6), (2.8), and (2.9) and the uniqueness of their solution.

4. The group property of α^t follows (for times small enough) from Eqs. (2.8) and (2.9) and the uniqueness of their solution; in particular, α^t is invertible and is therefore an automorphism of the algebra generated by $\mathscr{A}$ and $\mathscr{A}_\infty$.

5. As an automorphism, α^t preserves the norms and therefore, since the length of the time interval for which α^t is constructed only depends upon the norm of the Cauchy data (Theorem 3.4), the same construction can be repeated for a sequence of intervals of equal length; α^t_V converges therefore ultrastrongly for any t, the limit satisfies Eqs. (2.8) and (2.9), and all the results hold for all times. QED

We remark that the stability of $\mathscr{F}^+$ under the transpose of α^t, which is equivalent to $\mathscr{F}^+$-ultrastrong continuity of α^t,[12] is the same as "existence of the Schrödinger picture" for the dynamics, a point which has been the subject of some debate in the literature (see refs. 13–18).

3. ABSTRACT DIFFERENTIAL EQUATIONS

In this section we give the proof of the results on analytic differential equations in C^* algebras needed in Section 2. Results and proofs are almost straightforward generalizations of their counterparts for analytic differential equations for variables in $\mathbb{C}^n$ (see, e.g., ref. 25). Theorem 3.4 holds in complete normed algebras, while Theorem 3.5 makes use of a C^* and Von Neumann structure in order to deal with sequences of Cauchy data which do not converge in the norm topology.

Definition 3.1. Given a Banach algebra, i.e., a complete normed algebra $\mathscr{B}$, a map $f: \mathscr{B} \mapsto \mathscr{B}$ is called analytic if it is given by a norm-limit of polynomials, uniformly on bounded sets.

More generally, consider the space $\mathscr{B}^n \equiv \mathscr{B} \times \cdots \times \mathscr{B}$. The space $\mathscr{B}^n$ is a Banach space with the norm $\|(x^1,..., x^n)\| = \sup_i \|x_i\|$.

Definition 3.2. A map $f \equiv (f_1,..., f_k)$: $\mathscr{B}^n \mapsto \mathscr{B}^k$ is called analytic if, for $i = 1 \cdots k$, f_i is a norm-limit of polynomials (in n variables), uniformly on bounded sets in $\mathscr{B}^n$.

Definition 3.3. A map f: $I \subset \mathbb{R} \times \mathscr{B}^n \mapsto \mathscr{B}^k$ is called analytic if it is a norm-limit of polynomials (in all the variables), uniform on the products of the interval I with the bounded sets of $\mathscr{B}^n$.

It follows from the above definitions that the composition of analytic functions is analytic and that norm limits of analytic functions, uniform on bounded sets, are analytic.

Consider now equations of the form

$$\frac{dX}{dt} = F(X(t)) \tag{3.1}$$

with $X \in \mathscr{B}^n$ and dX/dt defined by $\|\cdot\|\text{-}\lim_{\varepsilon \to 0}[X(t+\varepsilon) - X(t)]/\varepsilon$.

Theorem 3.4. For any $X_0 \in \mathscr{B}^n$, $t_0 \in \mathbb{R}$, Eq. (3.1), with F analytic, has a unique solution $X(t, X_0)$ such that $X(t_0, X_0) = X_0$, defined for $|t - t_0| < r$, r a positive constant depending on F and $\|X_0\|$; $X(t, X_0)$ is an analytic function of t and X_0.

Proof. The solution is controlled, by the fixed point method, as a norm limit of analytic functions of t and X_0 (see, e.g., ref. 25, Theorem 2.2.2). For fixed $t_0 \in \mathbb{R}$, let $\mathscr{A}_{r,b}$ be the space of analytic functions

$$Y: \quad [t_0 - r, t_0 + r] \times \mathscr{B}^n \to \mathscr{B}^n$$

satisfying $Y(t_0, X) = X$, with the norm

$$\|Y\|_{r,b} \equiv \sup_{|t - t_0| \leqslant r, \|X\| \leqslant b} \|Y(t, X)\|_{\mathscr{B}^n}$$

Since F is analytic, the composite function $F(Y(t, X))$ is analytic, for any Y in $\mathscr{A}_{r,b}$, and

$$T(Y)(t, X) \equiv X + \int_{t_0}^{t} F(Y(s, X))\, ds$$

is in $\mathscr{A}_{r,b}$, since the integral of an analytic function is analytic and $T(Y)(t_0, X) = X$. Equation (3.1) for the variable $Y(t)$, with $Y(t_0) = X$, is equivalent to

$$Y(t, X) = X + \int_{t_0}^{t} F(Y(s, X))\, ds \tag{3.2}$$

i.e., $Y = T(Y)$. Existence and analyticity of the solution of Eq. (3.2), for any X in $\mathscr{B}^n$, follow from T being a contraction of the spheres

$$S_b \equiv \{Y \in \mathscr{A}_{r,b} \colon \|Y(t, X) - X\|_{r,b} \leqslant b\}$$

for any positive b and $r = r(b)$ defined as follows: Since F is analytic, there exist constants $M(a)$, $L(a)$ such that

$$\|F(X)\| \leqslant M(a), \qquad \|F(X) - F(Y)\| \leqslant L(a)\, \|X - Y\|$$

$\forall X, Y$ with $\|X\| \leqslant a$, $\|Y\| \leqslant a$.

It follows that, for $Y(t, X) \in S_b$,

$$\|T(Y) - X\|_{r,b} = \sup_{|t - t_0| \leqslant r, \|X\| \leqslant b} \left\| \int_{t_0}^{t} F(Y(s, X))\, ds \right\| \leqslant r M(2b)$$

since $\|X\| \leqslant b$, $|s-t_0| \leqslant r$, and $Y \in S_b$ imply $\|Y(s, X)\| \leqslant 2b$. Furthermore,

$$\|T(Y)-T(Z)\|_{r,b} = \sup_{|t-t_0| \leqslant r, \|X\| \leqslant b} \left\| \int_{t_0}^{t} F(Y(s, X)) - F(Z(s, X)) \, ds \right\|$$
$$\leqslant rL(2b) \|Y-Z\|_{r,b}$$

and therefore T maps S_b into itself, and it is a contraction, if

$$r(b) < \min(L(2b)^{-1}, bM(2b)^{-1})$$

The same argument also proves that T is a contraction in the space of functions $Y(t, X)$ which are continuous in t, with the same norm, and this implies that the solution of Eq. (3.2) is unique. QED

Theorem 3.5. If $\mathscr{B}$ is a C^* algebra, τ an ultrastrong topology on $\mathscr{B}$, and $\bar{\mathscr{B}}^{\mathscr{F}}$ the Von Neumann algebra obtained from $\mathscr{B}$ by closure in the topology τ, then the solution $X(t, X_0)$ of Eq. (3.1) with F analytic (Theorem 3.4), depends continuous on X_0 in the ultrastrong topology τ (for the components X_i of X and X_{0i} of X_0).

$X(t, X_0)$ extends by τ-continuity to the unique solution of Eq. (3.1), for variables in the product of the Von Neumann algebras $\bar{\mathscr{B}}^{\mathscr{F}}$, defined by $X(t_0) = X_0$, $X_0 \in \bar{\mathscr{B}}^{\mathscr{F}}$.

Proof. For any fixed t, $X(t, X_0)$ is an analytic function of X_0, i.e., a norm-limit of polynomials in X_{0i}. Since products are (jointly) continuous in (any) ultrastrong topology,[24] all polynomials converge ultrastrongly when the variables converge ultrastrongly, and so do their norm limits. By taking ultrastrong limits and using the analyticity of F, one sees that Eq. (3.1) is still satisfied, with d/dt defined as an ultrastrong limit $\lim_{\varepsilon \to 0}(X(t+\varepsilon) - X(t))/\varepsilon$. Since, by taking ultrastrongly convergent nets which are bounded in norm, F extends to an analytic function on the product of the Von Neumann algebras $\bar{\mathscr{B}}^{\mathscr{F}}$, we can apply Theorem 3.4 to Eq. (3.1) for variables in $\bar{\mathscr{B}}^{\mathscr{F}}$ and obtain that, for initial conditions X_0 in $\bar{\mathscr{B}}^{\mathscr{F}_n}$, $X(t, X_0)$ is still analytic; its time derivative exists therefore as a norm-limit. QED

Remark. It follows from the proof of Theorem 3.4 that $X(t, X_0)$ depends continuously, with the norm of $\mathscr{B}^n$, upon F with the topology of norm convergence, uniform on bounded sets. Therefore all the results hold when the functions f and g which appear in Eq. (2.6) depend on the volume, provided they converge (in norm, uniformly on bounded sets) when $V \to \infty$, in fact, norm convergence of $X(t, X_0)$ for fixed X_0 and $V \to \infty$ implies strong convergence of X when also X_0 depends on the volume and converge strongly, and this implies the results of Theorem 2.4.

4. EQUIVALENCE OF STRATEGIES AND DISCUSSION

In this section we shall prove that the dynamics of the (generalized) MFSM constructed in Section 2 gives the same result for the correlation function at all times as the contruction based on the convergence of the correlation function ("strategy one" mentioned in the Introduction) for a large class of finite-volume states, including Gibbs states at finite and zero temperature.

We shall then discuss briefly the implications of our results for the following subjects:

1. Dynamics of the MFSM out of thermal equilibrium.
2. General features of the dynamics of systems with long-range interactions.
3. Symmetry breaking and generalized Goldstone theorem.
4. KMS condition.

In the treatment based on finite-volume correlation functions[1,2,9] one considers a ground or Gibbs state ω_V and takes the infinite-volume limit of

$$\omega_V(\alpha_V^{t_1}(A_1)\cdots\alpha_V^{t_n}(A_n)), \qquad A_i \in \mathscr{A} \tag{4.1}$$

We will prove that, for any (generalized) MFSM, correlation functions of the form (4.1) converge to

$$\omega(\alpha^{t_1}(A_1)\cdots\alpha^{t_n}(A_n)), \qquad A_i \in \mathscr{A} \tag{4.2}$$

with α^t constructed in Section 2, provided only that:

(i) The states ω_V are invariant under the permutation of lattice points, a property always satisfied, as a consequence of the symmetry of the Hamiltonian under all exchanges of lattice points, by Gibbs states (for ground states there are exceptions in the presence of degeneracy; see below).

(ii) They converge to ω on $\mathscr{A}$ as $V\to\infty$, i.e., the correlation functions at time 0 converge (and define the state ω).

Proposition 4.1. Let ω_V be a $*$-weakly convergent sequence of states over the algebra $\mathscr{A}$ (introduced in Section 1):

$$\omega_V(A)\to\omega(A) \qquad \forall A\in\mathscr{A}$$

and let each ω_V be invariant under the group of automorphisms of $\mathscr{A}$ generated by all exchanges of lattice points (Section 2). Let H_V be given by

Eq. (1.2) [in particular, (1.1)] and α_V^t by Eq. (1.3). Then ω belongs to $\mathscr{F}^+$ and

$$\omega_V(\alpha_V^{t_1}(A_1)\cdots\alpha_V^{t_n}(A_n))$$

converges for $V\to\infty$, $\forall A_1\cdots A_n\in\mathscr{A}$, $\forall t_1\cdots t_n$ real, to

$$\omega(\alpha^{t_1}(A_1)\cdots\alpha^{t_n}(A_n))$$

with α^t constructed in Section 2.

Proof. 1. ω is in $\mathscr{F}^+$ by Proposition 2.3.

2. By Theorem 2.4, $\alpha_V^t(M_i^\alpha)$ is a norm-limit, uniform in V, of polynomials in the variables M_i^β and $(1/|V|)\sum_{i\in V}M_i^\gamma$, and $\alpha^t(M_i^\alpha)$ is a norm-limit of the same polynomials in the variables M_i^β and M_∞^γ; by an approximation in norm (and the fact that the matrices M_i^β are a linear basis for the algebra of matrices at lattice point i), it is therefore sufficient to prove convergence of all the correlation functions, on ω_V, of the variables M_i^β and $(1/|V|)\sum_{i\in V}M_i^\gamma$ to the corresponding correlation function, on ω, of M_i^β and M_∞^γ; these are well defined, since ω is in $\mathscr{F}^+$.

3. Convergence of such correlation functions is controlled as in the proof of Proposition 2.3: by symmetry under exchanges of lattice points, it is enough to consider correlation functions of the form

$$\sum_{j_1\cdots j_n\in V}\frac{1}{|V|^n}\,\omega_V(M_{i_1}^{\alpha_1}\cdots M_{i_k}^{\alpha_k}M_{j_1}^{\beta_1}\cdots M_{j_n}^{\beta_n}) \tag{4.3}$$

with $i_1\cdots i_k$ all different. With an error of order $1/|V|$, as in Proposition 2.3, the terms with at least two equal lattice indices can be ignored; the above sum coincides then with

$$\omega_V(M_{i_1}^{\alpha_1}\cdots M_{i_k}^{\alpha_k}M_{j_1}^{\beta_1}\cdots M_{j_n}^{\beta_n}) \tag{4.4}$$

with lattice indices all different; by symmetry under all exchanges of lattice points, this is independent of the lattice indices and converges, by assumption, to

$$\omega(M_{i_1}^{\alpha_1}\cdots M_{i_k}^{\alpha_k}M_{j_1}^{\beta_1}\cdots M_{j_n}^{\beta_n}) \tag{4.5}$$

This coincides with

$$\omega(M_{i_1}^{\alpha_1}\cdots M_{i_k}^{\alpha_k}M_{\infty}^{\beta_1}\cdots M_{\infty}^{\beta_n}) \tag{4.6}$$

since expression (4.6) is the limit for $V \to \infty$ of

$$\sum_{j_1 \cdots j_n \in V} \frac{1}{|V|^n} \omega(M_{i_1}^{\alpha_1} \cdots M_{i_k}^{\alpha_k} M_{j_1}^{\beta_1} \cdots M_{j_n}^{\beta_n})$$

which differs from (4.5) by order $1/|V|$. QED

Proposition 4.1 is of some interest in order to compare the general strategies for the construction of the dynamics of systems with long-range interactions discussed in the Introduction; from its proof it also follows that the assumption of invariance of ω_V under permutations of lattice points can be considerably relaxed: all that is needed is that the limit state ω be in $\mathscr{F}^+$ and that correlation functions of local *and mean variables* over the volume V converge to those of local *and ergodic* variables.

It must be remarked that it is easy to construct, for models with degenerate ground state, e.g., for the antiferromagnetic Ising model, examples of ground states ω_V which converge, for $V \to \infty$, to states which are not in $\mathscr{F}^+$, i.e., states which give rise to representations where ergodic limits do not exist and α^t is not therefore even defined. Such examples are, however, exceptional in the following sense: if one considers states $\omega_V^\infty \equiv \lim_{\beta \to \infty} \omega_V^\beta$, with ω_V^β states at finite inverse temperature β (the states ω_V^∞ are invariant under exchanges of lattice points, but in general not pure), then their limit for $V \to \infty$ *is* in $\mathscr{F}^+$ and this implies, by definition of $\mathscr{F}^+$, that all states appearing in their central decomposition *are* in $\mathscr{F}^+$, apart from a set of zero measure.

We now turn to points 1–4 mentioned above.

1. On the basis of the construction given in Section 2, the dynamics of the MFSM can be discussed in a large class of representations of the (spin) algebra which do not in general decompose into factorial representations invariant under time evolution. This allows for a nontrivial dynamics of ergodic means and therefore provides models for the "dynamics of classical variables out of equilibrium." Such variables would be represented by time-independent c-numbers[24] in any factorial representation defined, as is the case for a dynamics constructed following strategy (i), by a state invariant under time translations. As a consequence of the continuity of α^t in the (ultra-) strong topology defined by $\mathscr{F}$ (Theorem 2.4), the dynamics of ergodic means describes the dynamics of mean variables over large finite volumes and therefore has *a direct physical meaning.*

2. The general characteristic features of the dynamics of systems with long-range interactions[11] are shared by MFSM:

(i) The removal of the volume cutoff requires a strong topology, which is essentially given by the class of representation (of the quasilocal algebra) where the equations of motion in infinite volume make sense.

(ii) Variables at infinity appear in the time evolution of local variables, giving rise to a nontrivial center of the algebra stable under time evolution.

(iii) In time-translation invariant factorial representations, the dynamics α^t is described by automorphisms α^t_π of an algebra with trivial center; the automorphisms α^t_π *depend* upon the representation. Here this algebra can be identified with $\mathscr{A}$; in fact, α^t acts in the algebra generated by $\mathscr{A}$ and $\mathscr{A}_\infty$ and the elements of $\mathscr{A}_\infty$ are represented by time-translation-invariant (complex) numbers in any time-translation-invariant factorial representation of $\mathscr{A}$.

3. As is easily seen in special models, spontaneous symmetry breaking (SSB) in MFSM is not accompanied in general by the absence of an energy gap, and the occurrence of this phenomenon in the BCS model[1] shows that this mechanism has nontrivial physical implications. SSB in MFSM follows in fact[11] the general mechanism of SSB in models with long-range interaction proposed in refs. 11 and 12: a generalized Goldstone theorem relates the energy spectrum at momentum going to zero [of the states obtained by applying, to a (space- and) time-translation invariant state, the density of charge which generates the symmetry] to the dynamics *under* α^t_π of the means of the order parameter over large regions of space. This dynamics is not trivial, since the symmetry of the (finite-volume) Hamiltonians implies the symmetry of α^t, but fixing the variables at infinity to c-numbers spoils the symmetry in α^t_π; the transformation properties of α^t_π determine the energy spectrum[26] and are decided by the transformation under the symmetry of the ergodic variables and by the values they take in π.

Since their dynamics only involves strictly local and infinitely delocalized (ergodic mean) variables, MFSM can be seen as prototypes for the mechanism of SSB with an energy gap; even more, from this point of view, they are a sort of "simplest effective model" for the description of the low-energy spectrum in the presence of SSB without energy gap.

We remark again that the symmetry properties of the Hamiltonian are lost if the dynamics is constructed in special representations; one has then a situation with *explicit*, rather then spontaneous, symmetry breaking. The two approaches give strongly different results[27] in the presence of "external fields": in presence of *spontaneous* symmetry breaking, the order parameters follow the symmetry-breaking terms, while an *explicit* breaking contributes to the determination of the order parameters; as explained in ref. 27, this mechanism applies to nontrivial models and problems; e.g., a formulation giving rise to a *spontaneous* breaking of axial $U(1)$ symmetry in QCD leads to the alignment of the θ parameter with the fermion mass term and therefore to the absence of strong CP violation.

4. For (spin) systems with short-range interactions, the states ω which arise as infinite-volume limits of finite-volume Gibbs states at inverse temperature β (or appearing in their central decomposition) are characterized[3,6] by the Kubo–Martin–Schwinger (KMS) condition: $\forall A, B \in \mathscr{A}$, $F^{\omega}_{A,B}(t) \equiv \omega(\alpha^t(A)B)$ is an analytic function of t in the strip $0 < \mathrm{Im}(t) < i\beta$, and

$$F^{\omega}_{A,B}(t) = F^{\omega}_{B,A}(t + i\beta) \tag{4.7}$$

For MFSM, if the infinite-volume dynamics is defined in terms of correlation functions, the KMS condition must be formulated[8] as an analyticity property of the infinite-volume correlation functions with respect to a time evolution which is not constructed independently; it is not therefore a condition *on the states*, on the basis of a *given* dynamics.

On the other hand, the results on the connection of the KMS condition with thermodynamical and stability properties (see ref. 6) are based, also for MFSM, on the KMS condition (4.7), where α^t is assumed to be constructed as a strong limit in some class of representations of the spin algebra.

Our results, in particular Proposition 4.1, imply that for all (generalized) MFSM, the construction of the dynamics based on correlation functions and that based on strong limits coincide, for a large class of finite-volume states, and this implies that the limits of finite-volume Gibbs states *are* always KMS, also in the sense of Eq. (4.7).

We stress that this result holds for the dynamics α^t, but *not* for dynamics constructed with reference to restricted classes of states, in particular not for a (fixed) α^t_π; a nontrivial center is again necessary in order to formulate the KMS condition as a relation between dynamics and states.

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